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Fuzzy $\mathbb{C}P^2$

G. Alexanian [a], A.P. Balachandran [a], G. Immirzi [b,*], B. Ydri [a]

[a] *Physics Department, Syracuse University, Syracuse, NY 13244, USA*
[b] *Dipartimento di Fisica, Università di Perugia and INFN, I-06100 Perugia, Italy*

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Abstract

Regularization of quantum field theories (QFTs) can be achieved by quantizing the underlying manifold (spacetime or spatial slice) thereby replacing it by a non-commutative matrix model or a "fuzzy manifold". Such discretization by quantization is remarkably successful in preserving symmetries and topological features, and altogether overcoming the fermion-doubling problem. In this paper, we report on our work on applying this procedure of the four-dimensional $\mathbb{C}P^2$ and its QFTs. $\mathbb{C}P^2$ is not spin, but spin$_c$. Its Dirac operator has many unique features. They are explained and their fuzzy versions are described.

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1. Introduction

We can find few fundamental physical models amenable to exact treatment. Approximation methods like perturbation theory are necessary and are part of our physics culture.

Among the important approximation methods for quantum field theories (QFTs) are strong coupling methods based on lattice discretization of underlying spacetime or perhaps its time-slice. They are among the rare effective approaches for the study of confinement in QCD and for non-perturbative regularization of QFTs. They enjoyed much popularity in their early days and have retained their good reputation for addressing certain fundamental problems.

One feature of naive lattice discretizations however can be criticized. They do not retain the symmetries of the exact theory except in some rough sense. A related feature is that

* Corresponding author. Tel.: +39-55-4796-405; fax: +39-55-4796-254.
E-mail address: jgp@dma.unifi.it (B. Ydri).

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topology and differential geometry of the underlying manifolds are treated only indirectly, by limiting the couplings to "nearest neighbors". Thus, lattice points are generally manipulated like a trivial topological set, with a point being both open and closed. The upshot is that these models have no rigorous representation of topological defects and lumps like vortices, solitons and monopoles. The complexities in the ingenious solutions for the discrete QCD θ-term [1] illustrate such limitations. There do exist radical attempts to overcome these limitations using partially ordered sets [2], but their potentials are yet to be adequately studied.

A new approach to discretization, inspired by non-commutative geometry (NCG), is being developed since a few years [4–14]. The key remark here is that when the underlying spacetime or spatial cut can be treated as a phase space and quantized, with a parameter $\hat{h}$ assuming the role of $\hbar$, the emergent quantum space is fuzzy, and the number of independent states per ("classical") unit volume becomes finite. We have known this result after Planck and Bose introduced such an ultraviolet cut-off and quantum physics later justified it. A "fuzzy" manifold is ultraviolet finite, and if the parent manifold is compact too, supports only finitely many independent states. The continuum limit is the semi-classical $\hat{h} \to 0$ limit. This unconventional discretization of classical topology is not at all equivalent to the naive one, and we shall see that it does significantly overcome the previous criticisms.

There are other reasons also to pay attention to fuzzy spaces, be they spacetimes or spatial cuts. There is much interest among string theorists in matrix models and in describing D-branes using matrices. Fuzzy spaces lead to matrix models too and their ability to reflect topology better than elsewhere should therefore evoke our curiosity. They let us devise new sorts of discrete models and are interesting from that perspective. In addition, it has now been discovered that when open strings end on D-branes which are symplectic manifolds, then the branes can [16] become fuzzy, in this way one comes across fuzzy tori, $\mathbb{C}P^N$ and many such spaces in string physics.

The central idea behind fuzzy spaces is discretization by quantization. It does not always work. An obvious limitation is that the parent manifold has to be even dimensional. (See however [15], for constructing fuzzy $RP^3/\mathbb{Z}_2$ and other non-symplectic manifolds, even or odd.) If it is not, it has no chance of being a phase space. But that is not all. Successful use of fuzzy spaces for QFTs requires good fuzzy versions of the Laplacian, Dirac equation, chirality operator and so forth, and their incorporation can make the entire enterprise complicated. The torus T^2 is compact, admits a symplectic structure and on quantization becomes fuzzy, or a non-commutative torus. It supports a finite number of states if the symplectic form satisfies the Dirac quantization condition. But it is impossible to introduce suitable derivations without escalating the formalism to infinite dimensions [17].

But we do find a family of classical manifolds elegantly escaping these limitations. They are the co-adjoint orbits of Lie groups. For semi-simple Lie groups, they are the same as adjoint orbits. It is a theorem that these orbits are symplectic [18]. They can often be quantized when the symplectic forms satisfy the Dirac quantization condition. The resultant fuzzy spaces are described by linear operators on irreducible representations (IRRs) of the group. For compact orbits, the latter are finite-dimensional. In addition, the elements of the Lie algebra define natural derivations, and that helps to find Laplacian and the Dirac operator. We can even define chirality with no fermion doubling and represent monopoles and instantons (see [4–9] and the first three papers in [13]). These orbits therefore are altogether well-adapted for QFTs.

Let us give examples of these orbits.

- S^2: This is the orbit of $SU(2)$ through the Pauli matrix σ_3 or any of its multiples $\lambda\,\sigma_3$ ($\lambda \neq 0$). It is the set $\{\lambda g \sigma_3 g^{-1} : g \in SU(2)\}$. The symplectic form is $j\,\mathrm{d}\cos\theta \wedge \mathrm{d}\phi$ with θ, ϕ being the usual S^2 coordinates [19]. Quantization gives the spin j $SU(2)$ representations.
- $\mathbb{C}P^2$: $\mathbb{C}P^2$ is of particular interest being of dimension four. It is the orbit of $SU(3)$ through the hypercharge $Y = 1/3\,\mathrm{diag}(1, 1, -2)$ (or its multiples):

$$\mathbb{C}P^2 : \{gYg^{-1} : g \in SU(3)\}. \tag{1.1}$$

  The associated representations are symmetric products of 3's or $\bar{3}$'s (see Section 2).
- $SU(3)/[U(1) \times U(1)]$: This six-dimensional manifold is the orbit of $SU(3)$ through $\lambda_3 = \mathrm{diag}(1, -1, 0)$ and its multiples. These orbits give all the IRRs containing a zero hypercharge state.

In the literature, there are several studies of the fuzzy physics of $\mathbb{C}P^1 = S^2$ [3–15], while there is also a rigorous and beautiful treatment of $\mathbb{C}P^2$ by Grosse and Strohmaier [14]. The present work develops an alternative formulation for $\mathbb{C}P^2$. It is close to earlier treatments of S^2 [12,13] and seems to generalize to other quantizable orbits. It is eventually equivalent to that of [14] as we show, so that the first study of $\mathbb{C}P^2$ is of that reference.

Throughout this paper, we treat $\mathbb{C}P^2$ as Euclidean spacetime even though the possibility of treating it as spacial slice is also available.

Section 2 explains the basic properties of $\mathbb{C}P^2$. We quantize it in Section 3 to produce the fuzzy $\mathbb{C}P^2$ (some technical details necessary to quantization are provided in the Appendix A). Functional integral quantization of tensorial fields can also be done as we show in Section 4 (although topological considerations would prefer a more elaborate approach; see especially the first and last papers of [13]). In non-commutative geometry (NCG), a central role is assumed by the (massless) Dirac operator. Section 5 reviews it for $S^2 = \mathbb{C}P^1$ while Section 6 studies our approach to it in detail for $\mathbb{C}P^2$. Analysis shows its equivalence to the Dirac–Kähler operator [14]. $\mathbb{C}P^2$ is not a spin, but a spin$_c$ manifold, and that has exotic consequences for the $SU(3)$ spectrum: left- and right-chiral modes transform differently under $SU(3)$. Section 7 studies the fuzzy analogue of the Dirac operator. This work is greatly facilitated by coherent states and star ($\star$) products. The necessary material, contained in [9,15], is reviewed and used to discretize the continuum material here for both $S^2 = \mathbb{C}P^1$ and $\mathbb{C}P^2$. Incidentally the $\star$ product is particularly useful for formulating fuzzy analogues of important continuum quantities like correlation functions.

$\mathbb{C}P^2$ is a surface in $\mathbb{R}^8$ described by an algebraic equation. Appendix A establishes the fuzzy version of this equation and in addition useful identities among $SU(3)$ generators.

Appendix B is pedagogical and explains why $\mathbb{C}P^2$ is not spin and why the $SU(3)$ spectrum of the spin$_c$ Dirac operator has exotic features.

2. On $\mathbb{C}P^2$

$\mathbb{C}P^2$ is a Kähler manifold describable in different ways. Thus, as mentioned before it is the orbit of $SU(3)$ through the hypercharge operator Y or its multiples (the group $SU(3)$ has eight generators t_i which satisfy $[t_i, t_j] = i f_{ijk} t_k$; the hypercharge is $Y = (2/\sqrt{3})t_8$; in the **3** representation the generators are $(1/2)\lambda_i$, where the λ_i are the eight Gell–Mann matrices). As the stability group of Y is $U(2)$

$$U(2)=\left\{\begin{pmatrix} u & 0 \\ 0 & \det u^{-1}\end{pmatrix}\in SU(3)\right\}, \tag{2.1}$$

we have that

$$\mathbb{C}P^2 = SU(3)/U(2). \tag{2.2}$$

As its name reveals, it is also a projective complex space or the space of C^1 subspaces in C^3. If $\xi\in C^3-\{0\}$, a point of $\mathbb{C}P^2$ is the equivalence class $\langle\xi\rangle=\langle\lambda\xi\rangle$ for all $\lambda\in C^1-\{0\}$. Choosing $\lambda=\left(\sum|\xi_i|^2\right)^{-1/2}$, we see that $\mathbb{C}P^2=\{\langle\xi\rangle=\langle\xi \mathrm{e}^{\mathrm{i}\theta}\rangle\colon \left(\sum|\xi_i|^2\right)=1\}$. Hence,

$$\mathbb{C}P^2 = S^5/U(1). \tag{2.3}$$

In (2.1), we can first quotient $SU(3)$ by $SU(2)$. That is just the above S^5. That is because $SU(3)$ acts on C^3 and transitively on its sphere $S^5=\{\zeta\in C^3 : \sum|\zeta_i|^2=1\}$. At $(1,0,0)\in S^5$, the stability group is $SU(2)$ showing the result. In this way, we see that

$$\mathbb{C}P^2 = [SU(3)/SU(2)]/U(1) = S^5/U(1). \tag{2.4}$$

The eight Gell–Mann matrices form a basis for the real vector space of traceless hermitian matrices $\{\sum\xi_i\lambda_i, \xi=(\xi_1,\dots,\xi_8)\in R^8\}$. So $\mathbb{C}P^2$ is a sub-manifold of R^8. There is a beautiful algebraic equation for this sub-manifold. It is this: let d_{ijk} be the totally symmetric $SU(3)$-invariant tensor defined by

$$\lambda_i\lambda_j=\tfrac{2}{3}\,\delta_{ij}+(\,d_{ijk}+if_{ijk})\,\lambda_k \tag{2.5}$$

Then

$$\xi\in\mathbb{C}P^2 \Leftrightarrow d_{ijk}\xi_i\xi_j=\text{constant}\times\,\xi_k. \tag{2.6}$$

A pleasant manner to demonstrate this result is as follows. The symmetric $SU(3)$ invariant product $\chi,\eta\to\chi\vee\eta$, $(\chi\vee\eta)_i := d_{ijk}\chi_j\,\eta_k$ can be rewritten in terms of traceless hermitian matrices M, N as

$$M\vee N=\tfrac{1}{2}\{M,N\}-\tfrac{1}{6}\mathrm{Tr}(\{M,N\}), \tag{2.7}$$

where

$$M=\sum\chi_j\lambda_j,\qquad N=\sum\eta_j\lambda_j. \tag{2.8}$$

For this product, $M=\delta\lambda_8$ fulfills

$$M\vee M=-\frac{\delta}{\sqrt{3}}M \tag{2.9}$$

This relation is valid for points on the entire orbit through $\delta\lambda_8$ by $SU(3)$ invariance of the $\vee$ product.

Conversely, relation (2.9) implies that M is in the orbit of λ_8. For we can diagonalize M by an $SU(3)$ transformation g while keeping (2.9). After scaling the diagonal $M_{\mathrm{D}}=gMg^{-1}$

to Δ to reduce $-(\delta/\sqrt{3})$ to 1, we have $\Delta \vee \Delta = \Delta$, $\Delta = \mathrm{diag}(a, b, -a, -b)$. Comparing the difference of the first two rows on both sides, we get $a - b = (a + b)(a - b)$. If $a = b$, then $\Delta = 3aY$. If $a \neq b$, then $a + b = 1$. Comparing the first row, we get $a^2 - a - 2 = 0$, or $a = 2$ or -1. So $\Delta = \mathrm{diag}(2, -1, -1)$ or $\Delta = \mathrm{diag}(-1, 2, -1)$. Both become proportional to Y after Weyl reflections, establishing the result.

3. Quantizing $\mathbb{C}P^2$

A particular approach to quantizing co-adjoint orbits was developed many years ago in [19]. According to that method we obtain fuzzy $\mathbb{C}P^2$ quantizing the Lagrangian

$$L = i\bar{N}\mathrm{Tr}(Yg(t)^{-1}\dot{g}(t)), \quad g(t) \in SU(3), \quad \bar{N} = \text{constant}, \quad Y = \frac{\lambda_8}{\sqrt{3}} \tag{3.1}$$

A point $\xi(t) \in \mathbb{C}P^2$ is related to $g(t)$ by $\xi(t)_i\lambda_i = g(t)Yg^{-1}(t)$, while on $\mathbb{C}P^2$ the symplectic form is $i\bar{N}d\mathrm{Tr}Yg^{-1}dg = -i\bar{N}\mathrm{Tr}[Y\, g^{-1}dg \wedge g^{-1}dg]$. Writing $g = \mathrm{e}^{\mathrm{i}\lambda_i\theta^i/2}$, for a Hamiltonian description we may take as phase space (local) coordinates the θ^i and their conjugates $\pi_i = (\partial L/\partial\dot{\theta}^i)$, but the Lagrangian being of first order the latter are all constraints. To simplify the constraints we define E_{ij} by $g^{-1}dg = (\lambda_j/2i)E_{ji}d\theta^i$, and use the variables $\Lambda_{iR} = -\pi_j(E^{-1})_{ji}$, which have Poisson brackets

$$\{\Lambda_{iR}, g\} = g\frac{\lambda_i}{2i}, \qquad \{\Lambda_{iR}, \Lambda_{jR}\} = f_{ijk}\Lambda_{kR} \tag{3.2}$$

and are therefore the generators of $SU(3)$ transformations on $g(t)$ acting on the right. In terms of these variables the constraints $\pi_i - i\bar{N}\mathrm{Tr}(Yg^{-1}(\partial g/\partial\theta^i)) \approx 0$ become

$$\Lambda_{iR} + \frac{\bar{N}}{\sqrt{3}}\delta_{i,8} \approx 0 \tag{3.3}$$

They are second class for $i = 4, \ldots, 7$, first class for $i = 1, 2, 3, 8$, corresponding to the fact that if $g(t) \to g(t)\mathrm{e}^{\mathrm{i}\lambda_i\theta(t)/2}$, $i = 1, 2, 3, 8$, then $L \to L - (\bar{N}/\sqrt{3})\dot{\theta}\delta_{i,8}$. Thus, for the generator Y_R for right hypercharge we have $Y_R \approx -(2/3)\bar{N}$, and the right "isospin" generators $I_{\alpha R}$, $\alpha = 1, 2, 3$ vanish, $I_{\alpha R} \approx 0$. We can make a first class set (classically equivalent to all the constraints) by adding to these constraints complex combinations of the second class constraints: $Y_R \approx -(2/3)\bar{N}$, $I_{\alpha R} \approx 0$ and for $\bar{N} \geq 0$, $\Lambda_{4R} - i\Lambda_{5R} \approx \Lambda_{6R} - i\Lambda_{7R} \approx 0$ and for $\bar{N} \leq 0$, $\Lambda_{4R} + i\Lambda_{5R} \approx \Lambda_{6R} + i\Lambda_{7R} \approx 0$.

These constraints can be realized on functions on $SU(3)$. As isospin singlets have hypercharge in integral multiples of 2/3, we find that $\bar{N} \in \mathbb{Z}$. With $\bar{N}$ fixed accordingly, the constraints together mean that for right action, we have highest weight isospin singlet states of hypercharge $-(2/3)\bar{N}$.

An IRR of $SU(3)$ is labeled by (n_1, n_2), $n_i \in \mathbb{N}$. It comes from the symmetric product of n_1 $\underline{3}$'s and n_2 $\underline{3}^*$'s: a tensor $T^{i_1,\ldots,i_{n_1}}_{j_1,\ldots,j_{n_2}}$ for (n_1, n_2) has n_1 upper indices, n_2 lower indices and is traceless, $T^{i_1 i_2,\ldots,i_{n_1}}_{i_1 j_2,\ldots,j_{n_2}} = 0$. Within an IRR, the orthonormal basis can be written as $|(n_1, n_2), I^2, I_3, Y\rangle$ where I^2, I_3 and Y are square of isospin, its third component and hypercharge.

Let $g \to U^{(n_1,n_2)}(g)$ define the representation (n_1, n_2) of $SU(3)$. Then the functions given by $\langle (n_1, n_2), I^2, I_3, Y|U^{(n_1,n_2)}(g)|(n_1, n_2), 0, 0, -(2/3)\bar{N}\rangle$ fulfill the constraints. By the Peter–Weyl theorem, their linear span

$$\sum \xi^{(n_1,n_2)}_{I^2,I_3,Y}\langle (n_1, n_2), I^2, I_3, Y|U^{(n_1,n_2)}(g)|(n_1, n_2), 0, 0, -\tfrac{2}{3}\bar{N}\rangle \tag{3.4}$$

gives all the functions of interest.

If $\bar{N} = N \geq 0$, that requires that $(n_1, n_2) = (N, 0)$. These are just the symmetric products of N $\underline{3}$'s. If $\bar{N} = -N \leq 0$, $(n_1, n_2) = (0, N)$ or we get the symmetric product of N $\underline{3}^*$'s. The representations that we get by quantizing the Lagrangian (3.1) are thus $(N, 0)$ or $(0, N)$.

For $\mathbb{C}P^2$, there are coordinate functions $\hat{\xi}_i$, where $\hat{\xi}_i(\xi) = \xi_i$. The $\sum \hat{\xi}_i\,\hat{\xi}_i$ is a constant function which we can take to be $\mathbb{I}$, the function with value one. On quantization, $\hat{\xi}_i$ become the operators constant $\times \Lambda^L_i$ which we also denote as $\hat{\xi}_i$. Since $\sum \Lambda^L_i \Lambda^L_i = C_2\mathbb{I}$, and $C_2 = (1/3)N^2 + N$ in $(N, 0)$ or $(0, N)$ (see Appendix A), their exact form is

$$\hat{\xi}_i = \frac{\Lambda^L_i}{\sqrt{\frac{1}{3}N^2 + N}}, \qquad \sum \hat{\xi}_i\hat{\xi}_i = \mathbb{I}. \tag{3.5}$$

So

$$[\hat{\xi}_i, \hat{\xi}_j] = \frac{i}{\sqrt{\frac{1}{3}N^2 + N}} f_{ijk}\hat{\xi}_k \tag{3.6}$$

and they commute in the large N limit.

It is a remarkable fact that $\hat{\xi}_i$ fulfill (2.6) for *any* N if $\hat{\xi}_i$'s belong to $(N, 0)$ or $(0, N)$. A proof that uses the creation–annihilation operator techniques of Grosse and co-workers [6–9] is given in Appendix A. The result is a "fuzzy" analog of the defining relation (2.6)

$$d_{ijk}\hat{\xi}_i\hat{\xi}_j = \frac{(N/3) + (1/2)}{\sqrt{(1/3)N^2 + N}} \times \hat{\xi}_k. \tag{3.7}$$

The algebra A generated by $\hat{\xi}_i$ is what substitutes for the algebra of functions $\mathcal{A} = C^\infty(\mathbb{C}P^2)$. By Burnside's theorem [20], it is the full matrix algebra in the IRR. Fuzzy $\mathbb{C}P^2$ is just the algebra A.

The following point, emphasized by [14] is noteworthy. If $f \in \mathcal{A}$, it has the partial-wave expansion

$$\begin{aligned} &f(\xi) = \sum f^n_{I^2,I_3,Y}\langle (n_1, n_2), I^2, I_3, Y|U^{(n_1,n_2)}(g)|(n_1, n_2), 0, 0, 0\rangle, \\ &\xi_\alpha\lambda_\alpha := g\lambda_8\, g^{-1}. \end{aligned} \tag{3.8}$$

The ket $|(n_1, n_2), 0, 0, 0\rangle$ exists only if $n_1 = n_2$ so that the sum in (3.8) can be restricted to $n_1 = n_2$. If $F \in A$, then F too has an expansion like (3.8), where the series is cut-off at $n = N$. That is because of the following. The $SU(3)$ Lie algebra $su(3)$ has two actions on F: $F \to L^L_\alpha F = \Lambda_\alpha F$ and $F \to -L^R_\alpha F = -F\Lambda_\alpha$. The derivation $F \to ad\,L_\alpha F = L^L_\alpha F - L^R_\alpha F = [\Lambda_\alpha, F]$ is the action which annihilates $\mathbb{I}$ and corresponds to the $su(3)$ action on $\mathbb{C}P^2$. As F transforms as $(N, 0)$ (for $\bar{N} \geq 0$ say) for Λ^L_α and as $(0, N)$ for $-\Lambda^R_\alpha$,

A decomposes into direct sum of IRRs: $(N,0)\otimes(0,N)=\oplus_{n=0}^{N}(n,n)$. If $\langle(n,n),I^2,I_3,Y\rangle$ furnishes a basis for (n,n), then $F=\sum_0^N F^n_{I^2,I_3,Y}|(n,n),I^2,I_3,Y\rangle$. Identifying this basis with the one in (3.8) for $n\leq N$, we see that F transforms like a function on $\mathbb{C}P^2$ with a terminating partial-wave expansion.

A more precise statement is as follows [14]. We can put a scalar product on $\mathcal{A}$ using the Haar measure on $SU(3)$ and complete $\mathcal{A}$ into a Hilbert space $\mathcal{H}$. On $\mathcal{H}$, elements $\mathcal{F}$ of $\mathcal{A}$ act as linear operators by pointwise multiplication. Let $\mathcal{H}_{(N,0)}$ be the subspace of $\mathcal{H}$ carrying the IRR $(N,0)$ and $P_{(N,0)}:\mathcal{H}\to\mathcal{H}_{(N,0)}$ the corresponding projector. Then we have a map $\mathcal{A}\to P_{(N,0)}\,\mathcal{A}\,P_{(N,0)}$; $\mathcal{F}\to P_{(N,0)}\,\mathcal{F}\,P_{(N,0)}$ which is onto A. Thus, elements of A approximate functions in a good sense.

4. Fuzzy scalar fields

Here we briefly indicate a certain fuzzy version of the free scalar field action. It is very natural and a generalization of fuzzy $\mathbb{C}P^1$ action proposed earlier [5–11]. Still certain less obvious actions based on cyclic cohomology have been proposed [13,15], they have distinct topological advantages and correct continuum limits as well.

The operators $ad\,L_i=L_i^L-L_i^R$ correspond to the $SU(3)$ generators for functions on $\mathbb{C}P^2$. A Laplacian for fuzzy $\mathbb{C}P^2$ is thus $ad\,L_i^2$. A scalar field ϕ is a polynomial in the fuzzy coordinate functions $\hat{\xi}_i$, so ϕ is just a matrix in A. The Euclidean action for ϕ is

$$S(\phi)=\text{constant}\times\mathrm{Tr}(\phi^+ ad\,L_i^2\phi),\qquad ad\,L_i\phi=[L_i,\phi]. \tag{4.1}$$

Let λ_K be the eigenvalue of the continuum operator for the IRR (K,K); [14] gives

$$\lambda_K=2K(K+1). \tag{4.2}$$

If N is the maximum K for the fuzzy space, then $ad\,L_i^2$ has the spectrum $\{\lambda_0,\lambda_1,\ldots,\lambda_N\}$, it is just the cut-off spectrum of the continuum Laplacian.

5. The Dirac operator on $S^2\simeq\mathbb{C}P^1$

This section is a warm up for what follows on $\mathbb{C}P^2$ next. It contains a partial-wave analysis for the eigenstates of the S^2 Dirac operator $\mathcal{D}$ which can be generalized to $\mathbb{C}P^2$.

Let

$$S^2=\{x\in R^3:\sum x_\alpha^2=1\}, \tag{5.1}$$

and $\hat{x}$ be the coordinate functions: $\hat{x}_\alpha(x)=x_\alpha$.

Then the Dirac operator is

$$\begin{aligned}&\mathcal{D}=\sigma_\alpha\mathcal{P}_{\alpha\beta}\,J_\beta,\qquad \mathcal{P}_{\alpha\beta}=\delta_{\alpha\beta}-\hat{x}_\alpha\hat{x}_\beta,\qquad J_\beta=\mathcal{L}_\beta+\frac{\sigma_\beta}{2},\\ &\mathcal{L}_\alpha=-i(\hat{x}\wedge\vec{\nabla})_\alpha.\end{aligned} \tag{5.2}$$

$\mathcal{P}$ projects the Pauli matrices σ_α to their tangent space components $\sigma_\alpha\mathcal{P}_{\alpha\beta}$. $\mathcal{L}_\beta$ and J_β are orbital and total angular momenta, respectively.

If $f \in \mathcal{A} = C^\infty(S^2)$, it has the partial-wave expansion

$$f(x) = \sum_{kM} f_M^k \langle kM|D^{(k)}(g)|k0\rangle \tag{5.3}$$

where $D^{(k)} : g \to D^{(k)}(g)$ define the angular momentum k IRR of $SU(2)$ and $g\sigma_3\, g^{-1} = \sigma \cdot x$. The action of $\mathcal{L}_\alpha$ on it is specified by

$$\mathcal{L}_\alpha \langle kM|D^{(k)}(g)|k0\rangle = -\langle kM|J_\alpha^{(k)} D^{(k)}(g)|k0\rangle \tag{5.4}$$

where $J_\alpha^{(k)}$ are angular momentum k $SU(2)$-generators.

$\mathcal{D}$ acts on $\mathcal{A} \otimes C^2 \equiv \mathcal{A}^2 = \{(a_{1/2}, a_{-1/2}) : a_\lambda \in C^\infty(S^2)\}$. It anticommutes with the chirality operator

$$\Gamma = \sigma \cdot \hat{x}. \tag{5.5}$$

We now find the eigenfunctions of Γ.

Following (5.3), we can define a function x_α on $SU(2)$ as follows. For $g \in SU(2)$, $x_\alpha(g)$ is defined by $g\sigma_3 g^+ = \sigma \cdot x(g)$. The $\sigma \cdot \hat{x}$ is now the chirality operator on $\mathcal{A}^2$ defined in the following way. The action of $\sigma \cdot \hat{x}$ on $D^{(k)}$ is specified by $[\sigma \cdot \hat{x} D^{(1/2)}](g) = \sigma \cdot x(g) D^{(1/2)}(g)$. We will henceforth often omit g in writing $x(g)$. Since $D^{(1/2)}(g) = g$, it follows that helicity ± 1 eigenfunctions of $\sigma \cdot \hat{x} = D^{(1/2)}\sigma_3 D^{(1/2)} - 1$ are

$$D^{(1/2)}_{\cdot,\pm 1/2} = (D^{(1/2)}_{1/2,\pm 1/2}, D^{(1/2)}_{-1/2,\pm 1/2}). \tag{5.6}$$

Here, $D^{(1/2)}_{\cdot,\pm 1/2} \equiv \hat{g}_{\cdot,\pm 1/2}$, $\hat{g}_{ij}$ being functions on $SU(2)$: $\hat{g}_{ij}(g) = g_{ij}$. They have the equivariance property

$$D^{(1/2)}_{\cdot,\pm 1/2}(g\mathrm{e}^{\mathrm{i}\sigma_3\theta}) = D^{(1/2)}_{\cdot,\pm 1/2}(g)\mathrm{e}^{\pm \mathrm{i}\theta}. \tag{5.7}$$

Unlike (5.7), elements of $\mathcal{A}^2$ and hence too its chirality ± 1 subspaces $(1 \pm \sigma \cdot \hat{x}/2)\mathcal{A}^2$ are invariant under $g \to g\mathrm{e}^{\mathrm{i}\sigma_3\theta}$. The expansion of elements of these subspaces using the above D's must thus have another D in each term transforming with the opposite phase to that in (5.7). Accounting for this fact, we can write for $a \in A^2$,

$$a = a^+ + a^-, \qquad a^\pm = (a^\pm_{1/2}, a^\pm_{-1/2}) \in \frac{1 \pm \sigma \cdot \hat{x}}{2}\mathcal{A}^2,$$
$$a^\pm_\lambda = \sum_{n,j} \xi_n^{j\pm} D^{(j)}_{n,\mp 1/2} D^{(1/2)}_{\lambda,\pm 1/2}, \quad \xi_n^{j\pm} \in \mathbb{C}. \tag{5.8}$$

Now orbital angular momentum $\mathcal{L}_\beta$ is not defined on the individual factors in (5.8). We must lift it to the operator $\mathcal{J}_\beta^L$ which acts on $D^{(j)}$ and $D^{(1/2)}$ in such a manner that $\hat{x}$ transforms like a vector; $\mathcal{J}_\beta^L$ are $SU(2)$ generators acting by left translation

$$[\mathrm{e}^{\mathrm{i}\theta_\beta \mathcal{J}_\beta^L} D_{ij}^{(1/2)}](g) = D_{ij}^{(1/2)}(\mathrm{e}^{-\mathrm{i}\theta_\beta\sigma_\beta 2} g) = [\mathrm{e}^{-\mathrm{i}\theta_\beta\sigma_\beta 2} g]_{ij}. \tag{5.9}$$

We now reinterpret J_β as

$$J_\beta = \mathcal{J}_\beta^L + \frac{\sigma_\beta}{2}. \tag{5.10}$$

Because of the transformation rule (5.9), we find,

$$J_\beta D^{(1/2)}_{\cdot,\pm} = 0. \tag{5.11}$$

So

$$\mathcal{D} D^{(j)}_{n,\mp 1/2} D^{(1/2)}_{\cdot,\pm 1/2} = [\mathcal{J}^L_\beta D^{(j)}_{n,\mp 1/2}][\sigma_\alpha \mathcal{P}_{\alpha\beta} D^{(1/2)}_{\cdot,\pm 1/2}]. \tag{5.12}$$

Further simplification can be achieved by writing

$$\sigma_\alpha \mathcal{P}_{\alpha\beta} = [\tfrac{1}{2}\sigma \cdot \hat{x}, [\tfrac{1}{2}\sigma \cdot \hat{x}, \sigma_\beta]] = D^{(1/2)}[\tfrac{1}{2}\sigma_3, [\tfrac{1}{2}\sigma_3, \sigma_\alpha]] D^{(1/2)-1} D^{(1)}_{\beta\alpha}. \tag{5.13}$$

and noticing that $-D^{(1)}_{\beta\alpha} \mathcal{J}^L_\beta = \mathcal{J}^R_\alpha$ are $SU(2)$ generators acting in the right of g

$$[\mathrm{e}^{\mathrm{i}\theta_\alpha \mathcal{J}^R_\alpha} f](g) = f[g \mathrm{e}^{\mathrm{i}\theta_\alpha \sigma_\alpha 2}] \tag{5.14}$$

($f : SU(2) \to \mathbb{C}$ being a function on $SU(2)$). Putting this in (5.12)

$$\mathcal{D} D^{(j)}_{n,\mp 1/2} D^{(1/2)}_{\cdot,\pm 1/2} = -[D^{(j)}_{n,n'} (J^{(j)}_\alpha)_{n',\mp 1/2}] D^{(1/2)}_{\cdot,\mu'} [\tfrac{1}{2}\sigma_3, [\tfrac{1}{2}\sigma_3, \sigma_\alpha]]_{\mu',\pm 1/2} \tag{5.15}$$

The summation on α can be restricted to $\pm$ corresponding to raising and lowering operators as the $\alpha = 3$ term vanishes in the last factor.

It follows that:

$$\begin{aligned}(\mathcal{D}a)_\lambda = -\Bigg\{ &\sum_{j,n} \xi^{j+}_n D^{(j)}_{n,1/2} (J^{(j)}_+)_{1/2,-1/2}\, D^{1/2}_{\lambda,-1/2} \\ &+ \sum_{j,n} \xi^{j-}_n\, D^{(j)}_{n,-1/2} (J^{(j)}_-)_{-1/2,+1/2}\, D^{(1/2)}_{\lambda,+1/2} \Bigg\}\end{aligned} \tag{5.16}$$

which can also be written in the “Dirac–Kähler” form [14]

$$\mathcal{D} \begin{bmatrix} \sum \xi^{j+}_n D^{(j)}_{n,-1/2} \\ \sum \xi^{j-}_n D^{(j)}_{n,+1/2} \end{bmatrix} = \begin{bmatrix} 0 & (J^{(j)}_-)_{-1/2,+1/2} \\ (J^{(j)}_+)_{+1/2,-1/2} & 0 \end{bmatrix} \begin{bmatrix} \sum \xi^{j+}_n D^{(j)}_{n,+1/2} \\ \sum \xi^{j-}_n D^{(j)}_{n,-1/2} \end{bmatrix} \tag{5.17}$$

The eigenvalues of $\mathcal{D}$ are $\epsilon(j+(1/2))$ with $\epsilon = \pm 1$, each with degeneracy $(2j+1)$, while the corresponding eigenfunctions have $\xi^{j\epsilon\pm}_n = c^{j\epsilon}_n \langle j, \mp(1/2); (1/2), \pm(1/2) | j - (1/2)\epsilon, 0\rangle$. Explicitly,

$$(a^{j\epsilon})_\lambda = \sum_n c^{j\epsilon}_n \langle j, n; \frac{1}{2}, \lambda | D(g) | j - \frac{\epsilon}{2}, 0\rangle \tag{5.18}$$

where additional superscripts have been added to the eigenfunction.

6. The Dirac operator on $\mathbb{C}P^2$

$\mathbb{C}P^2$ is not spin, but spin$_c$ [14,21]. This fact introduces serious differences between the $\mathbb{C}P^2$ Dirac operator and the Dirac operator for a spin manifold such as $\mathbb{C}P^1$ discussed last.

The $\mathbb{C}P^2$ Dirac operator and its fuzzy version have been treated in [14]. Here we develop an alternative approach which seems capable of generalization to other coset spaces.

Elsewhere [15], we plan to treat tensor analysis on $\mathbb{C}P^N$ and its fuzzy versions in the language of projective modules. Here we will summarize just some points relevant for us. The next section will give their fuzzy versions.

6.1. The projective module for tangent bundle and its complex structure

The generators $ad\,\lambda_i$ in the adjoint representation $Ad : g \to Ad\,g$ of $SU(3) = \{g\}$ have matrix elements $(ad\lambda_i)_{jk} = -2if_{ijk}$, where f_{ijk} is totally antisymmetric. We have the identity $[\lambda_i, \lambda_j] = 2if_{ijk}\lambda_k$ and a similar relation for $ad\,\lambda_i$. As hypercharge commutes with itself and isospin generators, it follows that $f_{8ij} = 0$, if i or $j = 1, 2, 3$ or 8. Thus, the tangent vectors to $\mathbb{C}P^2$ at $\xi^0 = (0, \ldots, 0, 1)$, or equivalently at $\lambda_8 = \lambda_i\xi_i^0$, are $ad\lambda_j$, $j = 4, 5, 6, 7$. The directions $ad\lambda_j$, $j = 1, 2, 3, 8$ are normal. At any other point $\xi_i\lambda_i = g\lambda_8 g^{-1} \in \mathbb{C}P^2$, the normals accordingly are $Ad\,g(ad\,\lambda_j)\,Ad\,g^{-1} := \xi_i^{(j)}ad\lambda_i$, $j = 1, 2, 3, 8$, where $\xi_i^{(8)} = \xi_i$. That means that $f_{ikl}\xi_k\xi_l^{(j)} = 0$. The four orthogonal directions $Ad\,g(ad\,\lambda_j)\,Ad\,g^{-1}$ ($j = 4, 5, 6, 7$) in the trace norm span the tangent space.

The eigenvalues of $ad\,Y$ ($Y = (1/\sqrt{3})\lambda_8$), and hence also of $(1/\sqrt{3})\xi_i ad\lambda_i$ are $\pm 1, 0$ corresponding to the mesons K, $\bar{K}$, η^0 and $\vec{\pi}$ in the flavor octet terminology. If $\chi^{(+)}$ is an eigenvector for eigenvalue $+1$, $(1/\sqrt{3})(\xi_i ad\lambda_i)\chi^{(+)} = \chi^{(+)}$, then $\xi_i^{(j)}\chi_i^{(+)} = (1/\sqrt{3})\xi_k^{(j)}(\xi_l^{(8)}ad\lambda_l)_{ki}\chi_i^{(+)} = 0$ from above where $j = 1, 2, 3, 8$. Hence, $\chi^{(+)}$ is a tangent at ξ. So is $\chi^{(-)}$ for eigenvalue -1. Hence, $\chi^{(\pm)}$ span the tangent space and the null space of $\xi_i ad\lambda_i$ spans the space of normals.

We can now present sections of the tangent bundle $T\mathbb{C}P^2$ as a projective module. Let $\mathcal{A}^8 = \mathcal{A} \otimes C^8 = \{(\hat{\xi}_1, \ldots, \hat{\xi}_8)\}$.

$$\tfrac{1}{3}(\hat{\xi}_i ad\lambda_i)^2 = \mathcal{P} \tag{6.1}$$

is a projector and $\mathcal{P}\mathcal{A}^8$ is seen to consist of the sections of tangent bundle from the above remarks.

The complex structure on $\mathbb{C}P^2$ can be thought of as a splitting of the tangent space $T_\xi\mathbb{C}P^2$ as the direct sum $T_\xi^{(+)}\mathbb{C}P^2 + T_\xi^{(-)}\mathbb{C}P^2$ for all $\xi \in \mathbb{C}P^2$ in a smooth manner . The tensor $\mathcal{J}$ of complex analysis at ξ is then $\pm i$ on $T_\xi^{(\pm)}\mathbb{C}P^2$.

In the language of projective modules , we must thus split $\mathcal{P}$ as the sum of two orthogonal projectors $\mathcal{P}^{(\pm)}$. The tensor $\mathcal{J}$ is $\pm i$ on $\mathcal{P}^{(\pm)}\mathcal{A}^8$, that is, $\mathcal{J} = i(\mathcal{P}^{(+)} - \mathcal{P}^{(-)})$. Hence, also $\mathcal{J}\mathcal{P} = \mathcal{P}\mathcal{J} = \mathcal{J}$.

$SU(3)$-covariance suggests the choice of $\mathcal{P}^{(\pm)}\mathcal{A}^8$ as eigenspaces of $(1/\sqrt{3})\hat{\xi}_i ad\lambda_i$ for eigenvalues ± 1. Hence,

$$\mathcal{P}^{(\pm)} = \frac{1}{2\sqrt{3}}\hat{\xi}_i ad\lambda_i\left(\frac{1}{\sqrt{3}}\hat{\xi}_j ad\lambda_j \pm 1\right). \tag{6.2}$$

As $ad\lambda_i$ is antisymmetric, we have that

$$\mathcal{P}^{(+)\mathrm{T}} = \mathcal{P}^{(-)}, \qquad \mathcal{J}^{\mathrm{T}} = -\mathcal{J}. \tag{6.3}$$

From $\mathcal{J}$, we can also write the Levi–Civita symbol in an $SU(3)$-covariant way. It is

$$\epsilon_{ijkl} = 3\mathcal{J}_{[ij}\mathcal{J}_{kl]},\ [\,]:\ \text{antisymmetrization}. \tag{6.4}$$

6.2. The gamma matrices

Since $\mathbb{C}P^2$ is a sub-manifold of R^8 , it is natural to start from the Clifford algebra on R^8. Let its basis be the 16×16 matrices $\hat{\gamma}_i$ $(i = 1, 2, \dots, 8)$ with the relations

$$\{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij}, \quad \hat{\gamma}_i^{\dagger} = \hat{\gamma}_i \tag{6.5}$$

The γ-matrices which will occur in the Dirac operator are not these, rather they will be 16×16 γ-matrices γ_μ with the same relations

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad \gamma_i^{\dagger} = \gamma_i \tag{6.6}$$

but which act by left multiplication on the algebra generated by $\hat{\gamma}_i$, that is on the algebra Mat_{16} of 16×16 matrices M. Thus,

$$\gamma_i M = \hat{\gamma}_i M. \tag{6.7}$$

The matrices of Mat_{16} have a scalar product $(M, N) = \mathrm{Tr}\,(M^{\dagger}N)$ for which $\gamma_i^{\dagger} = \gamma_i$.

The $\mathbb{C}P^2$ γ's are the tangent projections $\gamma_i\mathcal{P}_{ij}$. There are only four of them at each ξ which are linearly independent. We have to find a four-dimension subspace of Mat_{16} at each ξ on which they can act. If we fail in that, we will end up with more than one fermion.

We first find this subspace at ξ^0. At ξ^0, define the fermionic creation–annihilation operators

$$\hat{a}_1^{\dagger} = \tfrac{1}{2}(\hat{\gamma}_4 + i\hat{\gamma}_5), \quad \hat{a}_1 = \tfrac{1}{2}(\hat{\gamma}_4 - i\hat{\gamma}_5), \quad \hat{a}_2^{\dagger} = \tfrac{1}{2}(\hat{\gamma}_6 + i\hat{\gamma}_7),\ \ \hat{a}_2 = \tfrac{1}{2}(\hat{\gamma}_6 - i\hat{\gamma}_7). \tag{6.8}$$

$\hat{a}_\alpha^{\dagger}$ transform as (K^+, K^0), $\hat{a}_\alpha$ as $(K^-, \bar{K}^0)$. Let

$$|0\rangle = \hat{a}_1\hat{a}_2, \qquad |\alpha\rangle = \hat{a}_\alpha^{\dagger}|0\rangle \quad (\alpha = 1, 2), \qquad |3\rangle = \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}|0\rangle. \tag{6.9}$$

They span a four-dimensional space. The γ_α $(4 \le \alpha \le 7)$ act irreducibly on this space. If

$$a_1^{\dagger} = \tfrac{1}{2}(\gamma_4 + i\gamma_5), \qquad a_2^{\dagger} = \tfrac{1}{2}(\gamma_6 + i\gamma_7) \tag{6.10}$$

and their adjoints define their creation–annihilation operators, $|0\rangle$ is their vacuum state.

For an appropriate subspace at other points of $\mathbb{C}P^2$, we use the fact that $SU(3)$ acts transitively on $\mathbb{C}P^2$. Thus, we can regard $\xi \in \mathbb{C}P^2$ as a function on $SU(3)$ with value $\xi(g)$ at g via the relation $g\lambda_8 g^{-1} = \lambda_i\xi_{i(g)}$. Then $\xi^0 = \xi(e)$, $e =$ identity.

Now the algebra of $SU(3)$ can be realized using γ_i, the generators being

$$t_i^c = \frac{1}{4i} f_{ijk}\gamma_j\gamma_k \tag{6.11}$$

Their 16-dimensional representation can be split into $\underline{8} \oplus \underline{8}$ using the projectors $P_\pm = (1 \pm \hat{\gamma}_9)/2$, $\hat{\gamma}_9 = \hat{\gamma}_1\hat{\gamma}_2, \dots, \hat{\gamma}_8$. The γ_i transform as an $\underline{8}$ under the action $\gamma_i \to [t_j^c, \gamma_i]$ by derivation

$$[t_j^c, \gamma_i] := ad\, t_j\gamma_i = if_{jil}\gamma_l. \tag{6.12}$$

Let $T(g)$ be the image of g in the $SU(3)$ representation given by (6.12). $T(g)$ can act on Mat_{16} by conjugation according to $Ad\,T(g)M = T(g)MT(g)^{-1}$. The $ad\,t_i$ are the infinitesimal generators for the action $Ad\,T(g)$ of $SU(3)$.

The four-dimensional vector space at $g = e$ and its basis can be labelled as $V(e)$ and $\{|\nu; e\rangle, \nu = 0, \alpha, 3 : |\nu; e\rangle = |\nu\rangle\}$. The vector space and its basis at g are then

$$\begin{aligned} V(g) &= Ad\,T(g)V(e) = T(g)V(e)T(g)^{-1}, \\ |\nu; g\rangle &= Ad\,T(g)|\nu; e\rangle = T(g)|\nu; e\rangle T(g)^{-1}. \end{aligned} \tag{6.13}$$

It is on this vector space that $\gamma_i P_{ij}(\xi(g))$ acts by left-multiplication.

On the vector space $V(e)$, the $U(2)$ subgroup of $SU(3)$ acts by conjugation. From the particle physics interpretation of $\hat{a}_\alpha^+$, we see that $V(e)$ decomposes into the direct sum

$$(I = 0, Y = -2) \oplus (\tfrac{1}{2}, -1) \oplus (0, 0). \tag{6.14}$$

To see the $SU(3)$ representation content of $|\nu, g\rangle$, let us first focus on $|0; g\rangle$. $|0; e\rangle \equiv |0\rangle$ is bilinear and antisymmetric in the γ's and has $I = 0$, $Y = -2$. The action $T(g)$ preserves the number of γ's. Thus, its $SU(3)$-orbit is contained in the vector space spanned by the antisymmetric product of two γ's, that is, $\gamma_{ij} = (1/2)(\gamma_i\gamma_j - \gamma_j\gamma_i)$. This vector space transforms as $\underline{10} \oplus \underline{\bar{10}} \oplus \underline{8}$. Only $\underline{10}$ contains an $I = 0$, $Y = -2$ vector, namely Ω^-, thus $|0; g\rangle \in \underline{10} = (N_1 = 3, N_2 = 0)$.

A more explicit formula can be written. Let $|(3, 0); (I, I_3, Y); e\rangle$ be the basis of vectors, which are linear in γ_{ij} and transforms as $\underline{10}$. We have: $|(3, 0); (0, 0, -2); e\rangle \equiv |0; e\rangle$. Then

$$|0; g\rangle = Ad\,T(g)\,|0; e\rangle = |(3, 0); (I, I_3, Y); e\rangle D^{(3,0)}_{(I,I_3,Y);(0,0,-2)}(g) \tag{6.15}$$

where $D^{(3,0)} : g \to D^{(3,0)}(g)$ is the IRR $\underline{10}$ of $SU(3)$ and the basis is labelled by (I, I_3, Y).

We can analyze the $SU(3)$ content and write an explicit formula for every $|\nu; g\rangle$,[1] $|\alpha; e\rangle$ $(\alpha = 1, 2)$ has γ_i's and γ_{ijk}'s where we mean by γ_{ijkl} the totally antisymmetrized product of $\gamma_i, \gamma_j, \gamma_k, \dots, \gamma_l$. The γ_i transforms as an $\underline{8}$ or $(N_1 = 1, N_2 = 1)$ while γ_{ijk} transforms as $\underline{27} \oplus \underline{10} \oplus \underline{\bar{10}} \oplus \underline{8} \oplus \underline{1}$. We can take linear combination of γ_i and γ_{ijk} to form two new $\underline{8}$'s such that the $\underline{8}$ part of $|\alpha; e\rangle$ is in a single $\underline{8}$. Also $|\alpha; e\rangle$ has $I = 1/2$, $Y = -1$ and such a vector occurs only in $\underline{8}$, $\underline{10}$ and $\underline{27}$. Thus, $|\alpha; g\rangle$ $(\alpha = 1, 2)$ transforms as the direct sum

[1] The analysis of the $SU(3)$ representation content of $V(g)$ is due to Brian Dolan.

$\underline{8} \oplus \underline{10} \oplus \underline{27}$; calculating explicitely the coefficients we find

$$\begin{aligned}|\alpha; g\rangle = &\sqrt{\tfrac{3}{5}}\,|(1,1); (I, I_3, Y); e\rangle D^{(1,1)}_{(I,I_3,Y),(1/2,(3-2\alpha)/2,-1)}(g) \\ &- \tfrac{1}{2}|(3,0); (I, I_3, Y); e\rangle D^{(3,0)}_{(I,I_3,Y),(1/2,(3-2\alpha)/2,-1)}(g) \\ &+ \sqrt{\tfrac{3}{20}}|(2,2); (I, I_3, Y); e\rangle D^{(2,2)}_{(I,I_3,Y),(1/2,(3-2\alpha)/2,-1)}(g)\end{aligned} \tag{6.16}$$

There remains $|3; e\rangle$ with $I = Y = 0$. It is a linear combination of a constant, γ_{ij}, and γ_{ijkl}. (i) The constant part transforms as an $SU(3)$-singlet; (ii) γ_{ij} was treated above; (iii) γ_{ijkl} is $\underline{27} \oplus \underline{8}$. $U(2)$ singlets with $I = Y = 0$ are contained only in $SU(3)$ singlet $\underline{8}$ and $\underline{27}$ so that $|3; g\rangle$ transforms as $\underline{1} \oplus \underline{8} \oplus \underline{27}$, the $\underline{8}$ being a mixture of $\underline{8}$'s from γ_{ij}, γ_{ijkl}. Calculating the coefficients explicitely, we find,

$$\begin{aligned}|3; g\rangle = &-\tfrac{1}{2}|0; (0,0,0); e\rangle + \sqrt{\tfrac{3}{5}}\,|(1,1); (I, I_3, Y); e\rangle D^{(1,1)}_{(I,I_3,Y),(0,0,0)}(g) \\ &+ \sqrt{\tfrac{3}{20}}\,|(2,2); (I, I_3, Y); e\rangle D^{(2,2)}_{(I,I_3,Y),(0,0,0)}(g).\end{aligned} \tag{6.17}$$

6.3. *The Dirac operator*

We require of the $\mathbb{C}P^2$ Dirac operator $\mathcal{D}$ that it is linear in derivatives and anticommutes with the chirality operator Γ

$$\Gamma := -\tfrac{1}{4!}\epsilon_{ijkl}\gamma_i\gamma_j\gamma_k\gamma_l. \tag{6.18}$$

At $\xi = \xi^0$, $\Gamma = -\gamma_4\gamma_5\gamma_6\gamma_7$ and is $+1$ on $|0; e\rangle$ and $|3; e\rangle$, and -1 on $|\alpha; e\rangle (\alpha = 1, 2)$. Hence, $\Gamma = +1$ on $|0; g\rangle$, $|3; g\rangle$ and -1 on $|\alpha; g\rangle$ for all g. The former have even chirality and the latter have odd chirality.

Now $\gamma_i P_{ij}$ anticommutes with Γ, while the $SU(3)$ generators

$$J_i = \mathcal{L}_i + ad\, t_i^c, \qquad \mathcal{L}_i = -if_{ijk}\hat{\xi}_j \frac{\partial}{\partial \hat{\xi}_k} \tag{6.19}$$

commute with Γ. Hence,

$$\mathcal{D} = \gamma_i \mathcal{P}_{ij} J_j \tag{6.20}$$

anticommutes with Γ,

$$\{\Gamma, \mathcal{D}\} = 0, \tag{6.21}$$

and is a good choice for the Dirac operator.

$\mathcal{D}$ acts on $\mathcal{A} \otimes \mathrm{Mat}_{16}$. But there are only four tangent gammas at each $\xi(g)$, so we have to reduce $\mathcal{A} \otimes \mathrm{Mat}_{16}$ to $V(g)$ (in a n appropriate sense) at each $\xi(g)$. We can achieve this reduction as follows. The functions $\hat{\xi}$ are defined according to

$$\hat{\xi}(g) = T(g)\, t_8^c\, T^{-1}(g). \tag{6.22}$$

where the notation means that T and T^{-1} are to be evaluated at g. Hence, if $u \in U(2)$, the stability group of t_8^c, $\hat{\xi}(gu) = \hat{\xi}(g)$. This means that $\mathcal{A} \otimes \mathrm{Mat}_{16}$ consists of sections of the trivial $U(2)$-bundle over $\mathbb{C}P^2$. The same is the case for its left- and right-chiral projections

$$\Psi_\pm = \frac{1 \pm \Gamma}{2} \mathcal{A} \otimes \mathrm{Mat}_{16}. \tag{6.23}$$

But that is not the case for $|0; g\rangle$ and $|\alpha; g\rangle$. Under $g \to g\,u$, $|0; g\rangle$ transforms as an $SU(2)$ singlet with $Y = -2$ and $|\alpha; g\rangle$ transforms as an $SU(2)$ doublet with hypercharge $Y = -1$.

Let $\hat{g}$ denote the matrix of functions on $SU(3)$ with $\hat{g}_{ij}(g) = g_{ij}$, $g \in SU(3)$ ($\hat{g}$ is just a simplified notation for $D^{(1,0)}$). We regard elements of $\mathcal{A} \otimes \mathrm{Mat}_{16}$ as functions of g, invariant under the substitution $g \to g\,u$. Accordingly, let us also introduce the vectors $|a; \hat{g}\rangle$; $a = 0, \alpha, 3$ which at g are the vectors $|a; \hat{g}(g)\rangle = |a; g\rangle$. Note that on a function f on $SU(3)$, the left- and right-actions of $h \in SU(3)$ are $f \to h^{L,R} f$, where $(h^L f)(g) = f(h^{-1} g)$ and $(h^R f)(g) = f(gh)$.

Now consider, in the case of $|0, g\rangle$, the wave functions $D^{(N_1,N_2)}_{(I,I_3 Y)(0,0,2)}$. They exist only if $N_2 = N_1 + 3$. The combination

$$D^{(N,N+3)}_{(I,I_3,Y)(0,0,2)} |0, \hat{g}\rangle \tag{6.24}$$

is invariant under $g \to gu$ at each g and can form constituents of a basis for the expansion of functions in $\mathcal{A} \otimes \mathrm{Mat}_{16}$.

The remaining elements of a basis can be found in the same manner, being

$$\frac{1}{\sqrt{2}} D^{(N_1,N_2)}_{(I,I_3,Y)(1/2,-m,1)} |\tilde{m}, \hat{g}\rangle; \quad N_2 - N_1 = 0 \text{ or } 3, \quad D^{(N,N)}_{(I,I_3,Y)(0,0,0)} |3, \hat{g}\rangle \tag{6.25}$$

where $\widetilde{1/2}, -\widetilde{1/2}$ stand for $\alpha = 1, 2$ and m is summed over.

There is a subtlety we encounter at this point. We also came across it for S^2. "Orbital" $SU(3)$ momentum $\mathcal{L}_\alpha$ does not act on the individual factors in (6.24) and (6.25), which are functions on $SU(3)$ and not just $\mathbb{C}P^2$. It is thus necessary to lift them to operators $\hat{\mathcal{J}}^L_i$ which act on $\hat{g}$ in such a manner that when (6.22) is used, $\hat{\xi}$ transform under $SU(3)$ in the way desired: $\hat{\xi} \to h^L \hat{\xi}$. Thus, $\hat{\mathcal{J}}^L_i$ are generators of $SU(3)_L$, the left-regular representation, and the Dirac equation is to be reinterpreted with J_j replaced by

$$J_j = \hat{\mathcal{J}}^L_j + ad\, t^c_j. \tag{6.26}$$

Restricted to $\mathcal{A} \otimes \mathrm{Mat}_{16}$, (6.26) is the same as (6.20).

The $|\nu; g\rangle$ is $T(g)|\nu; e\rangle T(g)^{-1}$ so $|\nu; \hat{g}\rangle$ is $T|\nu; e\rangle T^{-1}$. Now $(h^L T)(g) = T\,(h^{-1} g)$. That is, $h^L(T\,|\nu; e\rangle T^{-1})(g) = T(h^{-1}) T(g)|\nu; e\rangle\, T(g)^{-1} T(h)$, which has the infinitesimal form $\hat{\mathcal{J}}^L_j (T\,|\nu; e\rangle T^{-1})(g) = -ad\, t^c_j\, (T(g)|\nu; e\rangle T(g)^{-1})$. We conclude that

$$J_j |\nu, \hat{g}\rangle = 0. \tag{6.27}$$

The expression for $\mathcal{P}$ is in (6.1). Using commutation relations, we can write

$$\gamma_j \mathcal{P}_{ji} = \tfrac{4}{3} [t^c_k \hat{\xi}_k, [t^c_j \hat{\xi}_j, \gamma_i]]. \tag{6.28}$$

Now $Ad\,T\,\gamma_j = T\,\gamma_j T^{-1} = \gamma_k Ad\,\hat{g}_{kj}$, where $Ad\,\hat{g}(g) = Ad\,g$ represents g in the octet representation, it is real and orthogonal. Hence,

$$\gamma_j \mathcal{P}_{ji} = \tfrac{4}{3} \{[Ad\,T][t^c_8, [t^c_8, \gamma_j]][Ad\,T^{-1}]\} Ad\,\hat{g}_{ij}. \tag{6.29}$$

Since $|\nu, \hat{g}\rangle = Ad\,T|\nu; e\rangle$, $Ad\,T^{-1}|\nu; \hat{g}\rangle = |\nu; e\rangle$; $[t_8^c, [t_8^c, \gamma_i]]$ consists only of tangent space γ's at e. The action of $\{.\}$ on $|\nu; \hat{g}\rangle$ is thus

$$Ad\,T[t_8^c, [t_8^c, \gamma_i]]Ad\,T^{-1}|\nu; \hat{g}\rangle = Ad\,T\{[t_8^c, [t_8^c, \gamma_i]]|\nu; e\rangle\}. \tag{6.30}$$

The action of $\mathcal{D}$ on typical basis vectors like (6.24) follows:

$$\mathcal{D}D^{(N,N+3)}_{(I,I_3,Y)(0,0,2)}|0; \hat{g}\rangle = \tfrac{4}{3}\{Ad\hat{g}_{ij}\mathcal{J}_i D^{(N,N+3)}_{(I,I_3,Y)(0,0,2)}\}Ad\,T[t_8^c, [t_8^c, \gamma_j]]|0; e\rangle. \tag{6.31}$$

The term in braces also has a considerable simplification. Since $(h^L f)(g) = f(h^{-1}g) = f(g(g^{-1}h^{-1}g)) = [(\hat{g}^{-1}\hat{h}^{-1}\hat{g})^R f](g)$, $-Ad\hat{g}_{ij}\mathcal{J}_i^L$ are the generators $\mathcal{J}_\lambda^R$ for the $SU(3)$ acting on the right of g, they have the standard commutation relations $[\mathcal{J}_i^R, \mathcal{J}_j^R] = if_{ijk}\mathcal{J}_k^R$. We thus find that

$$\mathcal{D}D^{(N,N+3)}_{(I,I_3,Y)(0,0,2)}|0; \hat{g}\rangle = -\tfrac{4}{3}\{\mathcal{J}_i^R D^{(N,N+3)}_{(I,I_3,Y)(0,0,2)}\}\,Ad\,T\,[t_8^c, [t_8^c, \gamma_i]]|0; e\rangle. \tag{6.32}$$

The general wave function for even and odd chiralities can be written as

$$\begin{gathered}\xi_j^{(i)}D^{(i)}_{jj'}|j''; \hat{g}\rangle; \quad j' = (0,0,2), (0,0,0); \quad i = N_1, N_2, \\ N_2 - N_1 = 3,\ \text{if } j' = (0,0,2) \quad \text{and} \quad N_2 = N_1,\ \text{if } j' = (0,0,0),\end{gathered} \tag{6.33}$$

$$\begin{gathered}\eta_b^{(i)}D^{(i)}_{bb'}|b''; \hat{g}\rangle; \quad b' = (1/2, (3-2\alpha)/2, 1); \\ i = N_1, N_2, \quad N_1 - N_2 = 0 \bmod 3.\end{gathered} \tag{6.34}$$

Here j'', b'' are the state vectors for γ's pairing with j', b' as in (6.32) and $\xi_j^{(i)}, \eta_b^{(i)} \in \mathbb{C}$ and repeated indices are summed. Since $\gamma_j\mathcal{P}_{ji}$ anticommutes with Γ, we can represent the effect of $\mathcal{D}$ on wave functions in terms of the off-diagonal block

$$\begin{pmatrix} 0 & d \\ d^+ & 0 \end{pmatrix}, \tag{6.35}$$

acting on

$$\begin{pmatrix} \xi_j^{(i)}D^{(i)}_{jj'} \\ \eta_b^{(i)}D^{(i)}_{bb''} \end{pmatrix} \tag{6.36}$$

The result is the equation of [14] for $m = 0$; [14] has also found the spectrum of $\mathcal{D}$.

The zero modes of $\mathcal{D}$ can be easily worked out from (6.32). When $j' = (0,0,0)$, i can be $(0,0)$ (but not otherwise), and in that case, $D^{(0,0)}$ is a constant and is annihilated by $\hat{\mathcal{J}}_i^R$. Hence, the index of $\mathcal{D}$ is 1 and the zero mode has even chirality, consistently with [14].

7. Quantization

$\mathcal{D}$ acts on a subspace of $\mathcal{A} \otimes \mathrm{Mat}_{16}$. We can thus conceive of a fuzzy Dirac operator D which acts on a subspace of $A \otimes \mathrm{Mat}_{16}$, A being obtained from $\mathcal{A}$ by restricting "orbital"

$SU(3)$ IRRs to (n, n), $n \leq N$. D is then obtained from $\mathcal{D}$ by projection to this subspace. $\mathcal{D}$ commutes only with the total $SU(3)$ Casimir J_i^2 and not with orbital $SU(3)$ Casimir $\mathcal{L}_i^2$. This causes edge effects distorting the spectrum of D for those states having (n, n) near (N, N) which $\mathcal{D}$ mixes with (n', n'), $n' \geq N$. This particular edge phenomenon does not occur for $S^2 = \mathbb{C}P^1$ where orbital angular momentum $\mathcal{L}_\alpha^2$ commutes with the Dirac operator. A way to eliminate such problems is suggested by the work of [6–9]. We introduce the cut-off not on the orbital Casimir, but on the *total* Casimir, retaining all states upto the cut-off. That seems the best strategy as it will give a fuzzy Dirac operator D with a spectrum exactly that of the continuum operator $\mathcal{D}$ upto the cut-off point, and which has chirality (chirality Γ of $\mathcal{D}$ commutes with J_i^2) and no fermion doubling. This approach is the same as the method adopted for S^2 in [6–9]. For S^2, the edge effect turned up as the absence of the $-E$ eigenvalue subspace for the maximum total angular momentum when the cut-off is introduced in orbital angular momentum, and attendant problems with chirality.

D being just a restriction of $\mathcal{D}$, we can continue to use (6.20) in calculation, just remembering the truncation of the spectrum. That means that the analysis in Section 6 can be used intact. In the final expressions like (6.35) and (6.36), i labels the IRR and the Dirac operator acts in subspace with fixed i. So the cut-off can be introduced on $i = N_1, N_2$.

7.1. Coherent states and star products

These have been treated in [9,15,22]. Here we summarize the main points so that we can outline the relation of wave functions like (6.24) and those based on matrices for fuzzy physics.

7.1.1. The case of $S^2 \simeq \mathbb{C}P^1$

Let us first consider $S^2 = \mathbb{C}P^1$ and its fuzzy versions. The algebra A is Mat_{2l+1}. $SU(2)$ acts on A on left and right with generators L_i^L and $-L_i^R$, and orbital angular momentum is $\mathcal{L}_i = L_i^L - L_i^R$. The spectrum of $\mathcal{L}^2$ is $K(K+1)$, $K = 0, 1, \ldots, 2l$. We can find a basis of matrices T_M^K diagonal in $\mathcal{L}^2$ and $\mathcal{L}_3$ (with eigenvalue M) and standard matrix elements for $\mathcal{L}_i$. A acts on a $(2l+1)$-dimensional vector space with the familiar basis $|l, m\rangle$. T_M^K are orthogonal, $K(K+1)$ and M being eigenvalues of $\mathcal{L}^2$ and $\mathcal{L}_3$

$$(T_M^K, T_{M'}^{K'}) := \mathrm{Tr}\, T_M^{K\dagger} T_{M'}^{K'} = \text{constant} \times \delta_{KK'}\delta_{MM'}. \tag{7.1}$$

The above suggests that there is a way to regard A as "functions" on S^2 with angular momenta cut-off at $2l$. Such functions are also represented by the linear span of spherical harmonics Y_{KM}, $K \leq 2l$. We want to clarify the relation of Y_{KM}'s to the matrices T_M^K in A.

Towards this end, let us introduce the coherent states

$$|g; l\rangle = U^{(l)}(g)|l, l\rangle \tag{7.2}$$

induced from the highest weight vector $|l; l\rangle$. The $g \to U^{(K)}(g)$ is the angular momentum K IRR of $SU(2)$. Note the identity

$$|g\mathrm{e}^{\mathrm{i}(\sigma_3/2)\theta}; l\rangle = \mathrm{e}^{\mathrm{i}l\theta}|g; l\rangle. \tag{7.3}$$

It is a theorem [22], that the diagonal matrix elements $\langle g; l|a|g; l\rangle$ completely determine the operator a. Further $\langle g\mathrm{e}^{\mathrm{i}(\sigma_3/2)\theta}; l|a|g\mathrm{e}^{\mathrm{i}(\sigma_3/2)\theta}; l\rangle = \langle g; l|a|g; l\rangle$ so that $\langle g|a|g\rangle$ depends only on

$$g\sigma_3 g^+ = \sigma \cdot x, \qquad \sum x_i^2 = 1; x \in S^2. \tag{7.4}$$

In this way, we have the map $A \to C^\infty(S^2)$, $a \to \tilde{a}$, where $\tilde{a}(x) = \langle g|a|g\rangle$. In this map, the image of T_M^K is Y_{KM} after a phase choice

$$\langle g; l|T_M^K|g; l\rangle = Y_{KM}(x). \tag{7.5}$$

For, under $g \to hg$, $x \to R(h)x$, where $h \to R(h)$ is the $SU(2)$ vector representation. Under this transformation, since

$$Y_{KM}(R(h)x) = D^{(K)}(h)_{MM'} Y_{KM'}(x) \tag{7.6}$$

and

$$T_M^K \to U(h)^{-1} T_M^K U(h) = D^{(K)}(h)_{MM'} T_{M'}^K, \tag{7.7}$$

where $h \to D^{(K)}(h)$ is the angular momentum K IRR of $SU(2)$ in a matrix representation, we have the proportionality of the two sides (7.5) and phase conventions fix the constant of proportionality.

The map $T_M^K \to Y_{KM}$ is an isomorphism at the level of vector spaces. It can be extended to the non-commutative algebra A by defining a new product on Y_{KM}'s, the star product. Thus, consider $\langle g; l|T_M^K T_N^L|g; l\rangle$. The functions Y_{KM} and Y_{LN} completely determine T_M^K and T_N^L, and for that reason also this matrix element. Hence, it is the value of a function $Y_{KM} * Y_{LN}$, linear in each factor, at x

$$\langle g; l|T_M^K T_N^L|g; l\rangle = [Y_{KM} * Y_{LN}](x). \tag{7.8}$$

The product $*$ here, the star product, extends by linearity to all functions with angular momenta $\leq 2l$. The resultant algebra is isomorphic to the algebra A.

The explicit formula for $*$ has been found by Prešnajder [9] (see also [15]). The image of $\mathcal{L}_\alpha a$ is just $-i(\vec{x} \wedge \vec{\nabla})_\alpha \tilde{a}$. We will use the same symbol $\mathcal{L}_\alpha$ to denote $-i(\vec{x} \wedge \vec{\nabla})_\alpha$. The $*$ product is covariant under the $SU(2)$ action in the sense that

$$\mathcal{L}_\alpha(\tilde{a} * \tilde{b}) = (\mathcal{L}_\alpha \tilde{a}) * \tilde{b} + \tilde{a} * (\mathcal{L}_\alpha \tilde{b}). \tag{7.9}$$

It depends on l and approaches the commutative product of $C^\infty(S^2)$ as $l \to \infty$.

Coherent states thus give an intuitive handle on the matrix representation of functions. But on S^2, we also have monopole bundles. Sections of these bundles for Chern class n are spanned by the rotation matrices $D_{mn}^{(j)}$, $j \geq |n|$. They have the equivariance property

$$D_{mn}^{(j)}(g\mathrm{e}^{\mathrm{i}(\sigma_3/2)\theta}) = D_{mn}^{(j)}(g)\mathrm{e}^{\mathrm{i}n\theta}. \tag{7.10}$$

How do we represent them by matrices?

In the first instance, let $n \geq 0$ and consider the coherent states

$$|g; l+n\rangle = U^{(l+n)}(g)|l+n, l+n\rangle, \quad |g; l\rangle = U^{(l)}(g)|l, l\rangle. \tag{7.11}$$

They span vector spaces V_{l+n} and V_l. We can consider the linear operators $\mathrm{Hom}(V_{l+n}, V_l)$ from V_{l+n} to V_l. They are $[2l+1] \times [2(l+n)+1]$ matrices in a basis of V_{l+n} and V_l, and have $U^{(l)}(g)$ acting on their left (with generators L_i^L) and $U^{(l+n)}(g)$ acting on their right (with generators $-L_i^R$). We can decompose $\mathrm{Hom}(V_{l+n}, V_l)$ under the "orbital" angular momentum group $U^{(l)} \otimes U^{(l+n)}$ (with generators $\mathcal{L}_\alpha = L_\alpha^L - L_\alpha^R$) into the direct sum $\oplus_{K=n}^{2l+n}(K)$ with the IRR K having the basis T_M^K, with $\mathcal{L}_3 T_M^K = M T_M^K$. As before, we choose T_M^K so that $\mathcal{L}_i$ follow standard phase conventions. T_M^K are orthogonal

$$\mathrm{Tr}(T_{M'}^{K'})^\dagger T_M^K = \text{constant} \times \delta_{K'K}\delta_{M'M}. \tag{7.12}$$

Now consider

$$\langle l; g|T_M^K|g; l+n\rangle. \tag{7.13}$$

It transforms in precisely the same manner as $D_{Mn}^{(K)}(g)$ under $g \to hg$ and $g \to g\mathrm{e}^{\mathrm{i}(\sigma_3/2)\theta/2}$ and hence after an overall normalization,

$$\langle l; g|T_M^K|g; l+n\rangle = D_{Mn}^{(K)}(g). \tag{7.14}$$

Thus, $\mathrm{Hom}(V_{l+n}, V_l)$ are fuzzy versions of sections of vector bundles for Chern class $n \geq 0$. For $n < 0$, they are similarly $\mathrm{Hom}(V_l, V_{l+|n|})$. This result is due to [23] (see also [6–9,14]). An explicit formulae for the fuzzy version of rotation matrices can be found in [9].

It is interesting that Chern class has a clear meaning even in this matrix model. It is $|V| - |W|$ for $\mathrm{Hom}(V, W)$, where $|V|$ and $|W|$ are dimensions of V and W.

There are two (inequivalent) fuzzy algebras acting on $\mathrm{Hom}(V, W)$. $\mathrm{Mat}_{|V|} := A_{|V|}$ acts on the right and $\mathrm{Mat}_{|W|} := A_{|W|}$ acts on the left, where now a subscript has been introduced on A. These left and right actions have their own $*$'s, call them $*_{|V|}$ and $*_{|W|}$: if $a \in A_V$, $b \in A_W$ and $\tilde{a}$ and $\tilde{b}$ are the corresponding functions, then

$$bT_M^K a \to \tilde{b} *_{|W|} Y_{KM} *_{|V|} \tilde{a} \tag{7.15}$$

under the map of $\mathrm{Hom}(V, W)$ to sections of bundles.

There is also a fuzzy analogue for tensor products of bundles. Thus, we can compose elements of $\mathrm{Hom}(V, W)$ and $\mathrm{Hom}(W, X)$ to get $\mathrm{Hom}(V, X)$

$$\mathrm{Hom}(V, X) = \mathrm{Hom}(W, X) \otimes_{A_{|W|}} \mathrm{Hom}(V, W). \tag{7.16}$$

Its elements are TS, $S \in \mathrm{Hom}(V, W)$, $T \in \mathrm{Hom}(W, X)$. Its Chern class is $|V| - |X|$. If $\tilde{T}$ and $\tilde{S}$ are the representatives of T and S in terms of sections of bundles, then $TS \to \tilde{T} * \tilde{S}$.

Tensor products $\Gamma_1 \otimes \Gamma_2$ of two vector spaces Γ_1 and Γ_2 over an algebra B are defined only if $\Gamma_1(\Gamma_2)$ is a right- and left-B module [24]. Hence, $\mathrm{Hom}(W', X) \otimes_{A_{|W|}} \mathrm{Hom}(V, W)$ is defined only if $W = W'$. So $\tilde{T} * \tilde{S}$ is rather different in its properties from the usual tensor product of bundle sections, in particular $\tilde{S} * \tilde{T}$ makes no sense if $V \neq X$.

We can now comment on the fuzzy form of (5.17). Elsewhere, the Watamura and co-workers [10,11] and following them [12,13], investigated the Dirac operator as acting on $A \otimes C^2 = A^2$, $A = \mathrm{Mat}_{2l+1}$. That led to rather an elaborate formalism because of the cut-off in orbital angular momentum. So as indicated earlier, it seems more elegant to cut-off total angular momentum at some value j_0.

We can now argue such a cut-off leads to the formalism of [5–9] and to supersymmetry. Thus, let $T^j_{n+} \in \mathrm{Hom}(V_{l+1/2}, V_l)$ with the transformation property

$$U^{(l)}(g)^\dagger T^j_{n+} U^{(l+1/2)}(g) = D^{(j)}_{nn'}(g) T^j_{n'+} \tag{7.17}$$

So $j \leq 2l + 1/2$ and $j_0 = 2l + 1/2$. Then

$$D^j_{n+}(g) = \langle g; l | T^j_{n+} | g; l + 1/2\rangle \tag{7.18}$$

and an overall constant of proportionality has been set equal to 1 by suitably scaling T^j_{n+}. The subscript + indicates helicity (see Eq. (5.8)).

For helicity +, but for same j_0, we have to consider $T^j_{n-} \in \mathrm{Hom}(V_l, V_{l+1/2})$, with

$$U^{(l+1/2)}(g)^\dagger T^j_{n-} U^{(l)}(g) = D^{(j)}_{nn'}(g) T^j_{n'-}. \tag{7.19}$$

This is the formalism of [5–9]. As we have united $V^{(l)}$ and $V^{(l+1/2)}$, it is natural to consider $OSp(2, 1)$ or even $OSp(2, 2)$ *SUSY* as discovered first by Grosse et al. in the second paper of [6]. The action of the fuzzy Dirac operator D on $T^j_{n\pm}$ is merely the truncated form of (5.17)

$$DT_{n\pm} = -(J^{(j)}_{\pm})_{\pm(1/2),\mp(1/2)} T^j_{n\mp}, \quad j \leq 2l + 1/2. \tag{7.20}$$

Because of the mixing of l and $l + 1/2$, we have to reconsider the action of the matrix algebra A approximating $\mathcal{A} = C^\infty(S^2)$. Mat_{2l+1} acts on $T^j_{n+}(T^j_{n-})$ on the left (right) while Mat_{2l+2} acts on $T^j_{n+}(T^j_{n-})$ on the right (left). So it is best to regard fuzzy functions to act on left (say) of T^j_{n+} and right of T^j_{n-} as Mat_{2l+1}. This suggestion is slightly different from that of [5–9], where they regard the fuzzy algebra to be Mat_{2l+1} on T^j_{n+} and Mat_{2l+2} on T^j_{n-}, both acting on left. However, our proposal does not generalize to instanton (monopole) sectors.

We can restore spin parts to fuzzy wave functions. The spin wave functions for helicity $\pm$ are $T^{1/2}_{m\pm}$. So the two components of the total fuzzy wave functions for helicity $\pm$ are

$$\sum \xi^{j\pm}_n T^j_{n\mp} T^{1/2}_{\lambda\pm}, \qquad \xi^{j\pm}_n \in \mathbb{C}, \quad \lambda = 1, 2. \tag{7.21}$$

The Dirac operator D is given by the truncated version of (5.16)

$$\begin{aligned} &D_{\lambda\lambda'}\left\{\sum \xi^{j+}_n T^j_{n-} T^{1/2}_{\lambda'+} + \sum \xi^{j-}_n T^j_{n+} T^{1/2}_{\lambda'-}\right\}_m \\ &\quad = -\left\{\sum \xi^{j+}_n T^j_{n+} (J^{(j)}_+)_{+1/2,-1/2}\right\}\left\{T^{1/2}_{\lambda-}\right\} \\ &\qquad - \left\{\sum \xi^{j-}_n T^j_{n-} (J^{(j)}_-)_{-1/2,+1/2}\right\}\left\{T^{1/2}_{\lambda+}\right\}, \quad j \leq 2l + 1/2, \end{aligned} \tag{7.22}$$

$J^{(j)}_\alpha$ being the angular momentum j images of $(\sigma_\alpha/2)$.

7.1.2. *The case of* $\mathbb{C}P^2$

Coherent states for $\mathbb{C}P^2$ can be defined using highest weight states. For IRR (3, 0), we can pick the highest weight state with $I = I_3 = 0$, $Y = -2/3$, namely the λ-quark:

$|0, 0, -2/3\rangle \equiv |0, 0, -2/3; (3, 0)\rangle$. Then if $g \to U^{(3,0)}(g)$ defines the IRR, $|g; (3, 0)\rangle = U^{(3,0)}(g)|0, 0, -2/3; (3, 0)\rangle$. For the IRR $(N, 0)$, we can simply replace $|0, 0, -2/3; (3, 0)\rangle$ by its N-fold tensor product $|0, 0, -2/3; (3, 0)\rangle \otimes |0, 0, -2/3; (3, 0)\rangle \otimes \ldots \otimes |0, 0, -2/3; (3, 0)\rangle = |0, 0, -2N/3; (N, 0)\rangle$ and set

$$|g; (N, 0)\rangle = U^{(N,0)}(g)|0, 0, -\tfrac{2}{3}N; (N, 0)\rangle. \tag{7.23}$$

For $(0, N)$, we can use the $\bar{\lambda}$-quark state $|g; (0, 3)\rangle = U^{(0,N)}(g)|0, 0, +2/3; (0, 3)\rangle$ and its tensor product states.

The development of ideas now keep following $S^2 = \mathbb{C}P^1$. Full details can be found in [15]. General theory confirms that the map $a \to \tilde{a}$ from matrices in the $(N, 0)[(0, N)]$ IRR to functions on $\mathbb{C}P^2$, defined by $\tilde{a}(\xi) = \langle (N, 0); g|a|g; (N, 0)\rangle (\tilde{a}(\xi) = \langle (0, N); g|a|g; (0, N)\rangle)$ is one-to-one so that a $*$ product on $\tilde{a}$'s exists. In this map, the $SU(3)$ generators $\mathcal{L}_i$ acting on $\tilde{a}$ become the corresponding $\mathbb{C}P^2$ $SU(3)$ operators $-if_{ijk}\hat{\xi}_j(\partial/\partial\hat{\xi}_k)$. We shall use the same symbol $\mathcal{L}_i$ for these operators too. The orbital $SU(3)$ action is compatible with $*$ in the sense that $\mathcal{L}_i(\tilde{a} * \tilde{b}) = (\mathcal{L}_i\tilde{a}) * \tilde{b} + \tilde{a} * (\mathcal{L}_i\tilde{b})$. Irreducible tensor operators of $SU(3)$ are well studied [28]. With their help, fuzzy analogues of D-matrices can be constructed, as also sections of $U(1)$ and $U(2)$ bundles.

The fuzzy $\mathbb{C}P^2$ Dirac operator is the cut-off version of (6.32). It can be put in a matrix form as in (6.35) and (6.36). We omit the details: the necessary group theory is already to be found in [14] while the rest is routine.

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Appendix A. Fuzzy $\mathbb{C}P^2$ as "fuzzy" algebraic variety

Here we will provide the derivation of Eq. (3.7) and derive therefrom the expressions for the quadratic Casimir operator C_2 in $(N, 0)$ and $(0, N)$ representations.

The symmetric representations $(N, 0)$ of $SU(3)$ that appear in our $\mathbb{C}P^2$ study can be constructed using three creation operators $a_i^\dagger$ and their adjoints a_i. We have

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad i, j = 1, 2, 3. \tag{A.1}$$

For the representations $(0, N)$, we need three more creation operators $b_i^\dagger$ and their adjoints b_j. We concentrate below on $(N, 0)$, the treatment of $(0, N)$ being similar.

The $SU(3)$ generators are $\Lambda_a = a^\dagger t_a a,\ t_a = (1/2)\lambda_a$. They fulfill

$$[\Lambda_a, \Lambda_b] = if^{abc}\,\Lambda_c. \tag{A.2}$$

The Hilbert space $\mathcal{H}_{(N,0)}$ for $(N, 0)$ is spanned by states of the form

$$|n_1, n_2, n_3\rangle = a_1^{\dagger n_1} a_2^{\dagger n_2} a_3^{\dagger n_3} |0\rangle, \quad n_1 + n_2 + n_3 = N,$$
$$a_i |0\rangle = 0, \quad i = 1, 2, 3. \tag{A.3}$$

The dimension of this space is $(1/2)(N + 1)(N + 2)$.

Using the definition $d_{abc} = 2\,\mathrm{Tr}(t^a\{t^b, t^c\})$ the left-hand side of (2.6) becomes

$$d_{abc}\Lambda_b\Lambda_c = 2\,\mathrm{Tr}(t^a\{t^b, t^c\})\Lambda_b\Lambda_c \tag{A.4}$$

$$d_{abc}\Lambda_b\Lambda_c = 2t^a_{ij}(t^b_{jk}t^c_{ki} + t^c_{jk}t^b_{ki})\tfrac{1}{4}a^\dagger_m t^b_{mn}a_n a^\dagger_p t^c_{pq}a_q \tag{A.5}$$

The similar expression for the quadratic Casimir C_2 is

$$\Lambda_b\Lambda_b = a^\dagger_m t^b_{mn}a_n a^\dagger_k t^b_{kl}a_l \tag{A.6}$$

Taking advantage of the Fierz identity

$$\sum_\alpha (t^\alpha)_{ij}\,(t^\alpha)_{kl} = \tfrac{1}{2}\delta_{il}\delta_{jk} - \tfrac{1}{6}\delta_{ij}\delta_{kl} \tag{A.7}$$

to reduce the summations over the b and c indices, after a somewhat tedious, but straitforward, computation one gets

$$\Lambda_b\Lambda_b = \tfrac{1}{2}a^\dagger_m a_n a^\dagger_n a_m - \tfrac{1}{6}a^\dagger_m a_m a^\dagger_n a_n, \tag{A.8}$$

$$d_{abc}\Lambda_b\Lambda_c = 2t^a_{\alpha\beta}[\tfrac{1}{4}a^\dagger_l a_\beta a^\dagger_\alpha a_l - \tfrac{1}{6}a^\dagger_m a_m a^\dagger_\alpha a_\beta - \tfrac{1}{6}a^\dagger_\alpha a_\beta a^\dagger_l a_l + \tfrac{1}{4}a^\dagger_\alpha a_k a^\dagger_k a_\beta]. \tag{A.9}$$

where summation over the repeated indices is assumed.

At this point we have to use the fact that these operators act on the special states that belong to $\mathcal{H}_{(N,0)}$. For the states in (A.3), one has

$$\sum_i a^\dagger_i a_i |n_1, n_2, n_3\rangle = (n_1 + n_2 + n_3)|n_1, n_2, n_3\rangle = N|n_1, n_2, n_3\rangle. \tag{A.10}$$

From this and (A.1) we have the value of the quadratic Casimir $C_2 : C_2 = (1/3)N^2 + N$. Using the fact that t^a's are traceless, we find that the right-hand side of (A.9) when acting on the states from $\mathcal{H}_{(N,0)}$ becomes

$$d_{abc}\Lambda_a\Lambda_b = 2t^a_{\alpha\beta}\left[\frac{N}{6} + \frac{1}{4}\right]a^\dagger_\alpha a_\beta = \left(\frac{N}{3} + \frac{1}{2}\right)\Lambda_a. \tag{A.11}$$

Since $a^\dagger_i a_j$ transforms like $(N, 0) \otimes (0, N)$, Λ^L_i also fulfills (A.11). With $\hat{\xi}_i = \Lambda^L_i/\sqrt{(N^2/3) + N}$, (3.7) follows.

There are identical results for the $(0, N)$ representations. The proofs only involve replacing $a^\dagger_i$ and a_j by $b^\dagger_i$ and b_j in the preceding discussion.

Appendix B. Why $\mathbb{C}P^2$ is not spin

It is a standard result that $\mathbb{C}P^2$ does not admit a spin structure, but does admit a spin$_c$ structure. We plan to explain this result here adapting an argument of Hawking and Pope [21]. The reasoning shows that $\mathbb{C}P^N$ for any even $N \geq 2$ is not spin whereas it is spin if N is odd.

The obstruction to the $\mathbb{C}P^2$ spin structure comes from non-contractibile two-spheres in $\mathbb{C}P^2$. Since $\mathbb{C}P^2 \simeq SU(3)/U(2)$, $\pi_2(\mathbb{C}P^2) = \pi_1[U(2)] = \mathbb{Z}$. Also $\pi_1(\mathbb{C}P^2) = \{0\}$ so that Hurewitz's theorem [25] leads to $H^2(\mathbb{C}P^2, \mathbb{Z}) = \mathbb{Z}$. Its mod 2 reduction is $H^2(\mathbb{C}P^2, \mathbb{Z}_2) = \mathbb{Z}_2$. The absence of spin structure means that the tangent bundle is associated with the non-trivial element of $\mathbb{Z}_2$.

Consider a continuous map g of the square $\{(s,t) : 0 \leq s; t \leq 1\}$ into $SU(3)$ which obeys the following conditions (Fig. 1):

$$g(s,0) = g(0,t) = g(1,t) = \text{ identity } \mathbf{1}, \tag{B.1}$$

$$g(s,1) = \mathrm{e}^{\mathrm{i}\pi s(\lambda_3+\sqrt{3}\lambda_8)}. \tag{B.2}$$

The curve $g : (s,1) \to g(s,1)$ is a loop in $U(2) = \{\text{stability group of } \xi^0\}$ not contractible to identity while staying within $U(2)$. It is the generator of $\pi_1(U(2))$ and is associated with non-abelian $U(2)$ monopoles [26]. But since $\pi_1(SU(3) = \{0\}$, g can be defined smoothly in the entire square.

Now $U(2)$ being the stability group of ξ^0 is contained in the tangent space group $SO(4)$ at ξ^0. If $x = (x_\mu : \mu = 1,2,3,4)$ is a tangent vector at ξ^0, we can map it to a 2×2 matrix $M(x) = x_4 + i\vec{\tau} \cdot \vec{x}$ (τ_i = Pauli matrices) with the reality property $M(x)^* = \tau_2 M(x) \tau_2$. $SO(4) = [SU(2) \times SU(2)]/\mathbb{Z}_2$ acts on $M(x)$ according to $M(x) \to h_1 M(x) h_2^\dagger$, $h_i \in SU(2)$ preserving the reality property and the determinant $\det M(x) = \sum x_\mu^2$, and hence induces an $SO(4)$ transformation on x. $U(2)$ is imbedded in this $SO(4)$, acting on $M(x)$ as follows: $M(x) \to h_1 M(x) \mathrm{e}^{-\mathrm{i}\tau_3\theta}$.

The spin group $SU(2) \times SU(2) = \{(h_1,h_2)\}$ is a two-fold cover of $SO(4)$. The inverse image of $U(2)$ in $SU(2) \times SU(2)$ is $SU(2) \times U(1)$, also a two-fold cover of $U(2)$. In this cover the loop $g : (s,1) \to g(s,1)$ becomes $s \to (\mathrm{e}^{\mathrm{i}\pi s\tau_3}, \mathrm{e}^{\mathrm{i}\pi s\tau_3})$. It is no longer a loop,

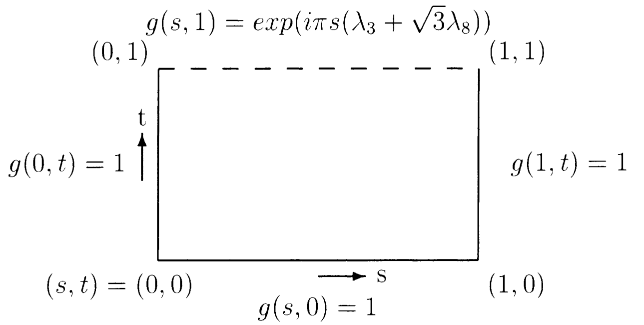

Fig. 1.

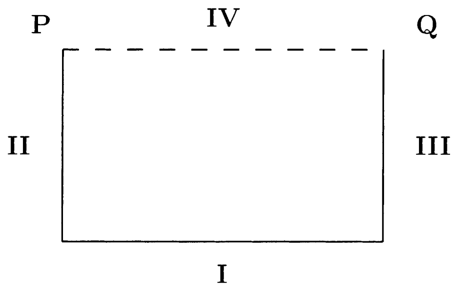

Fig. 2.

but runs from $(\mathbb{I}, \mathbb{I})$ to $(-\mathbb{I}, -\mathbb{I})$. It is this that obstructs the spin structure, as the following reasoning encountered in [21] shows.

Let $SU(3) \to \mathbb{C}P^2 \simeq SU(3)/U(2)$ be the map $h \in SU(3) \to h\lambda_8 h^{-1} = \lambda_\alpha \xi_\alpha$. $U(2)$ here has generators $\lambda_i\,(i = 1, 2, 3)$ and λ_8. This map takes the entire boundary of the square $\{g(s,t)\}$ to ξ^0 and the square itself to a two-sphere S^2.

Now the tangent space at ξ^0 of $\mathbb{C}P^2$ is spanned by the four $SU(3)$ Lie algebra directions $K^+, K^0, \bar{K}^0, K^-$ (in a complex basis). If we write $\mathbb{C}P^2$ as $\{h\lambda_8 h^{-1}\}$, a basis of tangents (a frame) at ξ^0 is $\lambda_a(a = 4, 5, 6, 7)$. Clearly $g(s,t)\lambda_a g(s,t)^{-1}$ gives a frame at $g(s,t)\lambda_8 g(s,t)^{-1}$ of $\mathbb{C}P^2$. This gives us a rule for transporting this frame (and hence any frame) smoothly along curves over $S^2 \in \mathbb{C}P^2$. If $\{(s(\tau), t(\tau)), 0 \le \tau \le 1\}$ is a curve on the square, the transport of the frame along the curve $g(s(\tau), t(\tau))\lambda_8 g(s(\tau), t(\tau))^{-1}$ in S^2 is $g(s(\tau), t(\tau))\lambda_a g(s(\tau), t(\tau))^{-1}$. In this rule, for the three sides I, II, III (see Fig. 2), we have $g(s,t)\lambda_8 g(s,t)^{-1} = \lambda_8$ and $g(s,t)\lambda_a g(s,t)^{-1} = \lambda_a$, so that we are at ξ^0 with the frame held fixed. Along side IV, we are still at λ_8 or ξ^0, but we are rotating λ_a according to $\exp\{i\pi s(\lambda_3 + \sqrt{3}\lambda_8)\}\,\lambda_a \exp\{-i\pi s(\lambda_3 + \sqrt{3}\lambda_8)\}$, it is a 2π- rotation of the frame as s varies from 0 to 1.

If spinors can be defined on $\mathbb{C}P^2$, this transport of frames will consistently lead to their transport as well. Thus, along sides I, II, III, we should be able to pick a suitable constant spinor ψ. But then, along IV, as s increases to 1, we will arrive at Q with $-\psi$ as $(-\mathbf{1}, -\mathbf{1})$ of $SU(2) \times SU(2)$ flips the sign of a spinor. As we had ψ along III, we lose continuity at P and find that spinors do not exist for $\mathbb{C}P^2$.

It is possible to show that this conclusion is not sensitive to our choice of rule of transport of frames (that is, connection in the frame bundle).

The spin$_c$ structure is achieved by introducing an additional $U(1)$ connection for spinors which amounts to adding a hypercharge of magnitude 1. That would give an additional phase $\exp(i\pi\sqrt{3}\lambda_8 s)$ along IV and an extra minus sign at $s = 1$ canceling the above unwanted minus sign. Note that: (1) this connection and extra hypercharge cancels out for frames which contain a spinor and a complex conjugate spinor; (2) there is no vector bundle with this extra connection as its existence gives a contradiction just as does the existence of the spin bundle.

Let us see what all this means for $SU(3)$. Under $U(2)$, at ξ^0, the tangents transform as K's and $\bar{K}$'s, that is as the IRRs $(I, Y) = (1/2, 1)$ and $1/2, -1)$. From the way $M(x)$

transforms, we can see that Y corresponds to τ_3 where $\tau_\alpha/2$ are $SU(2)$ generators acting on the right of $M(x)$.

The $SU(2)\times SU(2)$ IRRs of the non-existent spinors are as follows: (i) left-handed spinors: $(1/2, 0)$; (ii) right-handed spinors: (0,1/2). The corresponding (I, Y) quantum numbers are thus: (i) left-handed spinors: (1/2,0); (ii) right-handed spinors: (0,1) and $(0, -1)$. The quantum numbers in the spin$_c$ case follows by adding an additional hypercharge which we can take to be -1: (1) left-handed spin$_c$: $(1/2, -1)$; (2) right-handed spin$_c$: $(0, 0)$ and $(0, -2)$. These are precisely the $U(2)$ quantum numbers of the representation space of tangent γ's in Section 6. The $SU(3)$ IRRs have to contain these $U(2)$ IRRs. They are not symmetric between left- and right-handed spinors.

The spin$_c$ structures are not unique. Thus, we have the freedom to add additional hypercharge $2n$ ($n \in Z$) to the spin$_c$ spinors, that is, tensor the spin$_c$ bundle with any $U(1)$ bundle. The choice of spin$_c$ in our text is natural for our Dirac operator.

On general $\mathbb{C}P^N$: $\mathbb{C}P^N$ for all odd N admits a spin structure whereas those for even N admit only a spin$_c$ structure [27]. We can understand this result too by pursuing the preceding arguments.

Let $Y^{(N+1)} = 1/(N+1)\,\mathrm{diag}\,(1, 1, \dots, 1, -N)$ be the $SU(N+1)$ “hypercharge”. The previous Y is $Y^{(3)}$. We can represent $\mathbb{C}P^N = SU(N+1)/U(N)$ as $\{hY^{(N+1)}h^{-1} : h \in SU(N+1)\}$, the stability group $\{u \in SU(N+1) : uY^{(N+1)}u^{-1} = Y^{(N+1)}\}$ being $U(N)$.

For all $N \geq 1$, the square of Figs. 1 and 2 and the map $g : (s, t) \to g(s, t) \in SU(N+1)$ can be constructed so that it is constant on sides I, II and III while $g : (s, 1) \to g(s, 1)$ gives a generator of $\pi_1(U(N))$. There is obstruction to spin structure if this loop when it acts on a frame at $Y^{(N+1)}$ rotates it by 2π, that is acts as the non-contractible loop of $SO(2N)$.

Let $(q_1, q_2, \dots, q_{N+1})$ be the “quarks” of $SU(N+1)$. The hypercharge $Y^{(N)}$ of $SU(N)$ acts as the generator $\bar{Y}^{(N)} = (1/N)\,(1, 1, \dots, -(N-1), 0)$ on these quarks. We can choose the loop $g : (s, 1) \to g(s, 1)$ according to

$$g(s,1) = \mathrm{e}^{\mathrm{i}(2\pi s/N)(N\bar{Y}^{(N)})}\,\mathrm{e}^{-\mathrm{i}(2\pi s/N)(N+1)Y^{(N+1)}} = \begin{bmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & 0 & . & 0 \\ 0 & . & . & 1 & 0 \\ 0 & . & . & \mathrm{e}^{-i2\pi s} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{i2\pi s} \end{bmatrix}. \tag{B.3}$$

The tangent vectors at $Y^{(N+1)}$ transform like $\bar{q}^{(i)}q^{(N+1)}$ and $\bar{q}^{(N+1)}q^{(i)} (1 \leq i \leq N)$. So under $g(s, 1)$,

$$\begin{aligned} &\bar{q}^{(i)}q^{(N+1)} \to \mathrm{e}^{\mathrm{i}2\pi s}\bar{q}^{(i)}q^{(N+1)}, \quad i \leq N-1, \\ &\bar{q}^{(i)}q^{(N+1)} \to \mathrm{e}^{\mathrm{i}4\pi s}\bar{q}^{(i)}q^{(N+1)}, \quad i = N, \\ &\bar{q}^{(N+1)}q^{(i)} \to \mathrm{e}^{-\mathrm{i}2\pi s}\bar{q}^{(N+1)}q^{(i)}, \quad i \leq N-1, \\ &\bar{q}^{(N+1)}q^{(i)} \to \mathrm{e}^{-\mathrm{i}4\pi s}\bar{q}^{(N+1)}q^{(i)}, \quad i = N. \end{aligned} \tag{B.4}$$

Each i gives a plane in $2N$ dimensions and each factor $\mathrm{e}^{\mathrm{i}2\pi s}$ in the first two lines gives a 2π-rotation.

Thus, we have a product of $(N-1)+2=(N+1)$ 2π-rotations. For N odd, they are contractible in $SO(2N)$, and for N even, they are not, showing the result we were after.

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